

# RANKIN-SELBERG $L$ -FUNCTIONS IN CYCLOTOMIC TOWERS, II

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**ABSTRACT.** We establish an asymptotic analogue of Mazur's conjecture in the non self-dual setting, with applications to bounding ranks of elliptic curves in abelian extensions of imaginary quadratic fields. We also reduce the remaining cases of small ring class conductor in the conjecture to a straightforward nonvanishing criterion for individual cyclotomic twists.

## CONTENTS

1. Introduction	1
2. Nonvanishing via $p$ -adic $L$ -functions	7
References	18

## 1. INTRODUCTION

This article is a continuation of the prequel article [16], on the topic of Rankin-Selberg  $L$ -functions in cyclotomic towers, and particularly the nonvanishing of their central values in families. The purpose of the present article is to deduce a much stronger result towards the conjecture posed in [16] using only the theory of  $p$ -adic  $L$ -functions. In particular, we establish an asymptotic analogue of Mazur's conjecture in the non self-dual setting, with the remaining cases of small ring class conductors being reduced to a straightforward nonvanishing criterion for individual cyclotomic twists (Conjecture 1.2 below). In any case, the main result presented here applies to show asymptotic bounds for ranks of elliptic curves in maximal abelian  $p$ -extensions of imaginary quadratic fields via the associated two-variable Iwasawa main conjecture divisibilities (Theorem 1.4 below).

Let  $K$  be an imaginary quadratic field of discriminant  $D < 0$  and associated quadratic character  $\omega$ . Fix a rational prime  $p$ . Let  $R_\infty$  denote the maximal abelian extension of  $K$  unramified outside of  $p$ . Hence,  $R_\infty$  can be described as the compositum of towers  $K[p^\infty]K(\mu_{p^\infty})$ , where  $K[p^\infty] = \bigcup_{n \geq 0} K[p^n]$  is the union of all ring class extensions of  $p$ -power conductor over  $K$ , and  $K(\zeta_{p^\infty}) = \bigcup_{n \geq 0} K(\zeta_{p^n})$  the extension obtained by adjoining to  $K$  all  $p$ -power roots of unity. The ring class tower  $K[p^\infty]$  is Galois over  $\mathbf{Q}$ , and of generalized dihedral type. Its Galois group over  $K$  is isomorphic as a topological group to a finite extension of  $\mathbf{Z}_p$ . The cyclotomic tower  $K(\zeta_{p^\infty})$  is Galois over  $\mathbf{Q}$  and abelian. Its Galois group over  $K$

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is isomorphic as a topological group to  $\mathbf{Z}_p^\times$ . Let us write  $\mathcal{G}$  to denote the Galois group  $\text{Gal}(R_\infty/K)$ , with  $\Omega$  the Galois group  $\text{Gal}(K[p^\infty]/K)$  of the dihedral part, and  $\Gamma$  the Galois group of the  $\text{Gal}(K(\zeta_{p^\infty})/K)$  of the cyclotomic part, so that  $\mathcal{G} \approx \Omega \times \Gamma$ . The torsion subgroup  $G_0$  of  $\mathcal{G}$  is finite, and corresponds to the Galois group  $\text{Gal}(L/K)$  of the maximal tamely ramified extension  $L$  of  $K$  contained in  $R_\infty$ . The quotient group  $G$  of  $\mathcal{G}$  mod  $G_0$  is then topologically isomorphic to  $\mathbf{Z}_p^2$ , with a corresponding decomposition into dihedral and anticyclotomic parts denoted by  $G \approx \Omega \times \Gamma$ .

We consider the set  $\mathfrak{X}$  of finite order characters  $\mathcal{W}$  of  $\mathcal{G}$ , which by class field theory can be identified with the set of finite order Hecke characters of  $K$  unramified outside of  $p$  and infinity. We shall commit an abuse of notation throughout in always making such an identification implicitly. It is clear from the description of  $\mathcal{G}$  given above that any such character  $\mathcal{W}$  decomposes uniquely into a product of characters  $\mathcal{W} = \rho\psi$ , where  $\rho$  is a finite order character of the dihedral Galois group  $\Omega$ , and  $\psi$  is a finite order character of the cyclotomic Galois group  $\Gamma$ . The reader should note that each of these cyclotomic characters  $\psi$  takes the form  $\psi = \chi \circ \mathbf{N}$ , where  $\psi$  is some uniquely determined Dirichlet character of  $p$ -power conductor, and  $\mathbf{N}$  is the norm homomorphism on ideals of  $K$ . The decomposition  $\mathcal{G} \approx G \times G_0$  also induces a unique factorization of any such character  $\mathcal{W}$  into a product of characters  $\mathcal{W} = \mathcal{W}_0 \mathcal{W}_w$ , where  $\mathcal{W}_0$  is a tamely ramified character of the Galois group  $G_0$ , and  $\mathcal{W}_w$  is a wildly ramified character of the Galois group  $G$ . Thus, given a character  $\mathcal{W}$  of the set  $\mathfrak{X}$ , we shall always take these factorizations

$$\mathcal{W} = \rho\psi = \rho\chi \circ \mathbf{N} = \mathcal{W}_0 \mathcal{W}_w$$

to be fixed. We shall also assume throughout that  $\mathcal{W}$  has  $p$ -power conductor, writing  $X \subset \mathfrak{X}$  to denote the subset of such characters.

A classical construction due to Hecke associates to any character  $\mathcal{W}$  in  $\mathfrak{X}$  a theta series  $\Theta(\mathcal{W})$ , which is a modular form of weight 1, level  $\Delta$ , and Nebentypus  $\omega\chi^2$ . Here,  $\Delta = |D|c(\mathcal{W})^2$ , where  $c(\mathcal{W})$  denotes the (norm of) the conductor of  $\mathcal{W}$ . Moreover, this theta series  $\Theta(\mathcal{W})$  can be characterized as the inverse Mellin transform of the complex  $L$ -function  $L(s, \mathcal{W})$  associated to  $\mathcal{W}$ . Let us now fix a cuspidal Hecke newform  $f$  of weight 2, level  $N$ , and trivial Nebentypus. We write

$$f(z) = \sum_{n \geq 1} a_n(f) q^n$$

to denote its Fourier series expansion at infinity, where as usual  $z = x+iy$  denotes an element of the complex upper-half plane  $\mathfrak{H} = \{z \in \mathbf{C} : \Im(z) > 0\}$ , the coefficients  $a_n(f)$  are normalized so that  $a_1(f) = 1$ , and  $q^n = \exp(2\pi i n)$ . We consider the Rankin-Selberg  $L$ -function of  $f$  times any of the theta series  $\Theta(\mathcal{W})$  described above, normalized to have central value at  $s = 1/2$ , which we denote here by the symbol  $L(s, f \times \mathcal{W})$ . Hence,  $L(s, f \times \mathcal{W})$  can be described via its Dirichlet series expansion

$$L(s, f \times \mathcal{W}) = L^{(N)}(2s, \omega\chi^2) \sum_{\substack{n \geq 1 \\ (n, c(\mathcal{W}))=1}} \left( \sum_A \rho(A) r_A(n) \right) a_n(f) \chi(n) n^{-s},$$

which converges absolutely for  $\Re(s) \gg 1$ . Here,  $L(s, \omega\chi^2)$  denotes the usual Dirichlet series  $L(s, \omega\chi^2)$  of the character  $\omega\chi^2$ , with primes dividing  $N$  removed from its Euler product. We have also used the factorization  $\mathcal{W} = \rho\chi \circ \mathbf{N}$  of  $\mathcal{W}$  into dihedral and cyclotomic parts described above. Moreover, we have viewed the underlying

dihedral character  $\rho$  as a ring class character of the class group  $\text{Pic}(\mathcal{O}_c)$ , where  $\mathcal{O} = \mathbf{Z} + c\mathcal{O}_K$  is the  $\mathbf{Z}$ -order of conductor  $c = c(\rho)$  in  $K$ . And finally, given a class  $A$  in  $\text{Pic}(\mathcal{O}_c)$ , we have written  $r_A(n)$  to denote the number of ideals of absolute norm  $n$  in  $A$ . The classical Rankin-Selberg method shows that this  $L$ -function  $L(s, f \times \mathcal{W})$  has an analytic continuation to the complex plane, and moreover that its completed  $L$ -function

$$\Lambda(s, f \times \mathcal{W}) = (N\Delta)^s \Gamma_{\mathbf{R}}(s + 1/2) \Gamma_{\mathbf{R}}(s + 3/2) L(s, f \times \mathcal{W})$$

satisfies the functional equation

$$(1) \quad \Lambda(s, f \times \mathcal{W}) = \epsilon(s, f \times \mathcal{W}) \Lambda(1 - s, f \times \overline{\mathcal{W}}).$$

Here, we have written  $\Gamma_{\mathbf{R}}(s)$  to denote  $\pi^{-s/2} \Gamma(s/2)$ , with  $\epsilon(s, f \times \mathcal{W})$  the epsilon factor associated to  $\Lambda(s, f \times \mathcal{W})$ , and  $\overline{\mathcal{W}}$  to denote the contragredient character associated to  $\mathcal{W}$ . The epsilon factor at the central point  $\epsilon(1/2, f \times \mathcal{W})$  is a complex number of modulus one known as the *root number* of  $L(s, f \times \mathcal{W})$ . In this particular setting, one can see from the classical derivation of (1) via convolution that the root number  $\epsilon(1/2, f \times \mathcal{W})$  is given simply by  $-\omega \chi^2(N')$ , where  $N'$  denotes the prime-to- $p$  part of the level  $N$  of  $f$ , at least if  $N$  is prime to the discriminant  $D$  of  $K$ .

The  $L$ -function  $L(s, f \times \mathcal{W})$  for any character  $\mathcal{W}$  in  $\mathfrak{X}$  is said to be *self dual* if the coefficients in its Dirichlet series expansion are real-valued, or equivalently if its root number  $\epsilon(1/2, f \times \mathcal{W})$  takes values in the set  $\{\pm 1\}$ . This is well known to be the case for the  $L$ -functions  $L(s, f \times \mathcal{W})$  where  $\mathcal{W} = \rho$  is a dihedral or ring class character in the description given above. Moreover, if  $\mathcal{W} = \rho$  is such a character, then it is easy to see that the functional equation (1) relates the same completed  $L$ -function on either side, i.e. that  $\Lambda(s, f \times \mathcal{W}) = \Lambda(s, f \times \overline{\mathcal{W}})$ . This is a consequence of the well known fact that such characters (viewed as Hecke characters of  $K$ ) are equivariant with respect to complex conjugation (cf. [9, p. 384]). In this particular setting with  $\mathcal{W} = \rho$ , the condition that the root number  $\epsilon(1/2, f \times \mathcal{W})$  equal  $-1$  then imposes the vanishing of the associated central value  $L(1/2, f \times \mathcal{W})$ . To distinguish this particular case in all that follows, let us define a pair  $(f, \mathcal{W})$  to be *exceptional* if (i)  $\mathcal{W} = \rho$  is a ring class character and (ii) the root number  $\epsilon(1/2, f \times \mathcal{W})$  is  $-1$ . We then define a pair  $(f, \mathcal{W})$  for any given character  $\mathcal{W}$  in the set  $\mathfrak{X}$  to be *generic* if it is not exceptional.

The purpose of the present work, following that of the prequel [16], is to study the nonvanishing behaviour of the central values  $L(1/2, f \times \mathcal{W})$  in the generic setting. Such nonvanishing is predicted by the generalized conjecture of Birch and Swinnerton-Dyer via the associated context of two-variable main conjectures for elliptic curves, as we explain in some more detail below. To describe the expected behaviour, let us first recall the celebrated algebraicity theorem of Shimura [12], in particular as it applies to the central values  $L(1/2, f \times \mathcal{W})$  described above. Thus, let us write  $F = \mathbf{Q}(a_n(f))_{n \geq 0}$  to denote the extension of the rational number field  $\mathbf{Q}$  obtained by adjoining all of the normalized Fourier coefficients  $a_n(f)$  of  $f$ . Let us then write  $F(\mathcal{W})$  to denote the extension of  $F$  obtained by adjoining the values of  $\mathcal{W}$ . The main algebraicity theorem of Shimura [11] shows that the values

$$(2) \quad \frac{L(1/2, f \times \mathcal{W})}{\langle f, f \rangle}$$

lie in  $F(\mathcal{W})$ , where  $\langle f, f \rangle$  denotes the Petersson inner product of  $f$  with itself. In particular, these values (2) are algebraic. They are also Galois conjugate in the

following sense. Writing  $\mathcal{W} = \rho\chi \circ \mathbf{N} = \mathcal{W}_0\mathcal{W}_w$  as above, let us define  $P_{c,q;\mathcal{W}_0}$  to be the set of all such characters  $\mathcal{W}$ , where  $\rho$  is primitive of some conductor  $c$ ,  $\chi$  is primitive of some conductor  $q$ , and the tamely ramified part  $\mathcal{W}_0$  is fixed. The main result of Shimura [11] then implies that the value  $L(1/2, f \times \mathcal{W})$  vanishes for some character  $\mathcal{W} \in P_{c,q;\mathcal{W}_0}$  if and only if the value  $L(1/2, f \times \mathcal{W})$  vanishes for all characters  $\mathcal{W} \in P_{c,q;\mathcal{W}_0}$ . Equipped with this notion, we can associate to any character  $\mathcal{W}$  of  $\mathfrak{X}$  an associated Galois average,

$$\delta_{[\mathcal{W}]} = \delta_{c,q;\mathcal{W}_0} = |P_{c,q;\mathcal{W}_0}|^{-1} \sum_{\mathcal{W} \in P_{c,q;\mathcal{W}_0}} L(1/2, f \times \mathcal{W}).$$

Of course, if  $(f, \mathcal{W})$  is exceptional, then we can see from the functional equation (1) that the associated Galois average  $\delta_{[\mathcal{W}]}$  must vanish. In this setting, one studies instead the central values of the associated first derivatives  $L'(1/2, f \times \mathcal{W})$ . One can establish via the formulae of Gross-Zagier [5] and Zhang [22] an analogous notion of Galois conjugacy for these values, as explained in [16] (cf. also [9]). This leads us to define for  $k = 0$  or  $1$  the notion of a  $k$ -th Galois average  $\delta_{[\mathcal{W}]}^{(k)} = \delta_{c,q;\mathcal{W}_0}^{(k)}$ ,

$$\delta_{[\mathcal{W}]}^{(k)} = \delta_{c,q;\mathcal{W}_0}^{(k)} = |P_{c,q;\mathcal{W}_0}|^{-1} \sum_{\mathcal{W} \in P_{c,q;\mathcal{W}_0}} L^{(k)}(1/2, f \times \mathcal{W}).$$

Here,  $L^{(0)}(1/2, f \times \mathcal{W})$  is taken to denote the central value  $L(1/2, f \times \mathcal{W})$ , and  $L^{(1)}(1/2, f \times \mathcal{W})$  that of the derivative  $L'(1/2, f \times \mathcal{W})$ . In general, we expect that if the conductor of  $\mathcal{W}$  is sufficiently large, then  $\delta_{[\mathcal{W}]}^{(k)}$  does not vanish, where  $k$  is taken to be  $0$  or  $1$  according as to whether the pair  $(f, \mathcal{W})$  is generic or exceptional respectively. This expectation in the exceptional setting with  $k = 1$  can be deduced from the theorem of Cornut [3], as we shall explain in some more detail below. Our aim here is to establish this expectation asymptotically in the generic setting with  $k = 0$ , building on the analytic estimates of the prequel work [16], as well as giving asymptotic generalizations of the earlier works of Rohrlich [10], [9], and Vatsal [19] for the associated one-variable settings (i.e. with either  $\mathcal{W} = \rho$  ring class or  $\mathcal{W} = \psi = \chi \circ \mathbf{N}$  cyclotomic in the setup described above). The novelty of this work is that we shall use essentially no analysis in the proofs, but rather the existence of some associated  $p$ -adic  $L$ -functions to reduce the problem to previously established results in the self-dual setting via suitable (systematic) applications of the Weierstrass preparation theorem. Moreover, the method presented here allows us to give streamlined proofs of some of the works mentioned above (namely those of [10] and [19]), and appears to work in much greater generality. Here, we obtain the following main results, using the work of Cornut [3] with suitable algebraicity theorems to deduce the stated results in the exceptional setting. Using the existence of an associated two-variable  $p$ -adic  $L$ -function due to Hida [6] and Perrin-Riou [8], along with the more general nonvanishing results of Cornut-Vatsal [4] for self dual Rankin-Selberg  $L$ -functions over totally real fields, we obtain the following result.

**Theorem 1.1** (Theorem 2.7). *Fix an embedding  $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ . Assume that the eigenform  $f$  is  $p$ -ordinary in the sense that the image of its  $T_p$ -eigenvalue under this embedding is a  $p$ -adic unit. Assume additionally that  $p \geq 5$ , and that the prime to  $p$ -part of the level  $N$  of  $f$  is prime to the discriminant  $D$  of  $K$ . Fix a tamely ramified character  $\mathcal{W}_0$  of  $p$ -power conductor of  $G_0 \approx \text{Gal}(L/K)$ . Let  $n \geq 0$  be any integer. There exists an integer  $c(0) \geq 0$ , independent of choice of  $n$ , such that*

for each possible ( $p$ -power) dihedral or ring class conductor  $c \geq c(0)$ , the associated Galois average  $\delta_{c, p^n; \rho_0}^{(k)}$  does not vanish. Here,  $k = 0$  or  $1$  according as to whether the pair  $(f, \mathcal{W})$  is generic or exceptional respectively.

We conjecture that this results extends to all possible ( $p$ -power) ring class conductors  $c \geq 0$  in the generic setting with  $k = 0$  and  $n$  sufficiently large. More precisely, as we shall see in the discussion below using the Weierstrass preparation theorem, these remaining cases can be established via the following criterion.

**Conjecture 1.2.** *Fix a ring class character  $\rho = \rho_0 \rho_w$  in the set  $\mathfrak{X}$  of any given conductor  $c$ . Then, there exists a cyclotomic character  $\psi = \chi \circ \mathbf{N}$  in  $\mathfrak{X}$  such that the central value  $L(1/2, f \times \rho\psi)$  does not vanish.*

Observe that we have of course established this criterion in Theorem 2.7 above for  $c$  sufficiently large, i.e. for  $c \geq c(0)$ . It seems very likely that this criterion in the remaining cases of small ring class conductor  $c$  can be treated by certain averaging techniques that are beyond the scope of the present paper. We hope to take this up in a subsequent work. Anyhow, if we can establish the nonvanishing criterion of Conjecture 1.2 for a ring class character  $\rho = \rho_0 \rho_w$  of each possible conductor  $c < c(0)$  and given tamely ramified part  $\rho_0$ , then we can deduce from the discussion below the following full analogue of Mazur's conjecture in the non-self dual setting. Recall that we write  $X$  to denote the set of finite order characters of the Galois group  $\mathcal{G}$  having  $p$ -power conductor. Let  $X^{(0)}$  denote the subset of characters  $\mathcal{W}$  in  $X$  for which the pair  $(f, \mathcal{W})$  is generic, and  $X^{(1)}$  the subset of characters  $\mathcal{W}$  in  $X$  for which the pair  $(f, \mathcal{W})$  is exceptional.

**Corollary 1.3.** *For each choice of tamely ramified ring class character  $\mathcal{W}_0 = \rho_0$  of  $G_0$ , assume Conjecture 1.2 for one primitive ring class character  $\rho = \rho_0 \rho_w$  of each possible conductor  $c < c(0)$  in  $X^{(0)}$ . Then, for all but finitely many characters  $\mathcal{W}$  in  $X^{(k)}$ , the associated  $k$ -th Galois average  $\delta_{[\mathcal{W}]}^{(k)}$  does not vanish. Here,  $k = 0$  or  $1$  according as to whether the pair  $(f, \mathcal{W})$  is generic or exceptional respectively.*

An unconditional partial analogue of this result can also be established via the main estimate of the prequel work [16], i.e. for ring class characters  $\rho = \rho_0 \rho_w$  of conductors smaller than  $c(0)$ , where we do not specify the tamely ramified part  $\rho_0$ . The difficulty in extending this unconditional result to each individual  $\rho_0$  seems to be in the apparent algebraic independence of the associated twisted values  $L(1/2, f \times \rho_0 \rho_w \psi) / 8\pi^2 \langle f, f \rangle \in \overline{\mathbf{Q}}$ .

**Bounds for ranks of elliptic curves.** Let  $K_\infty$  denote the  $\mathbf{Z}_p^2$ -extension of  $K$ , so that  $G \approx \mathcal{G}/G_0$  above is identified with the Galois group  $\text{Gal}(K_\infty/K)$ . We obtain the following unconditional bounds for ranks of elliptic curves in the tower  $K_\infty/K$ , thanks to the two-variable main conjecture divisibility shown by Skinner-Urban [13] (cf. [7] or [18]). Let  $E$  be an elliptic curve of conductor  $N$  defined over  $\mathbf{Q}$ , without complex multiplication. We know by fundamental work of Wiles [20], Taylor-Wiles [14] and Breuil-Conrad-Diamond-Taylor [1] that  $E$  is modular, hence parametrized by a cuspidal newform  $f$  of weight 2, level  $N$  and trivial Nebentypus. This for instance allows us to identify the Hasse-Weil  $L$ -function  $L(E/K, \mathcal{W}, s)$  of  $E$  over  $K$  twisted by a finite order character  $\mathcal{W}$  of  $\mathcal{G}$  with the Rankin-Selberg  $L$ -function  $L(s, f \times \mathcal{W})$  described above, at least up to normalization factor. Let us now take  $\mathcal{O}$  to be the ring of integers of some finite extension of  $\mathbf{Q}_p$  containing the values of  $\mathcal{W}$ . Let  $\mathcal{O}[[G]]$  denote the  $\mathcal{O}$ -Iwasawa algebra of  $G$ , i.e. the completed group ring

of  $G$  with coefficients in  $\mathcal{O}$ , or equivalently the ring of  $\mathcal{O}$ -valued measures on  $G$ . As explained below (Theorem 2.1), there exists an element  $\mathcal{L}_p(f/K_\infty)$  in  $\mathcal{O}[[G]]$  that interpolates  $p$ -adically the algebraic values (2), or rather the images of these values under any fixed embedding  $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ . In particular, the specialization of this element to any finite order character  $\mathcal{W}$  of  $X$  vanishes if and only if the associated central value  $L(1/2, f \times \overline{\mathcal{W}})$  vanishes.

Let us now write  $\text{Sel}(E/K_\infty)$  to denote the  $p$ -primary Selmer group of  $E$  in the tower  $K_\infty/K$ , which fits into the exact descent sequence of discrete  $\mathcal{O}[[G]]$ -modules

$$(3) \quad 0 \longrightarrow E(K_\infty) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \text{Sel}(E/K_\infty) \longrightarrow \text{III}(E/K_\infty)(p).$$

Here,  $E(K_\infty)$  denotes the Mordell-Weil group of  $E$  over  $K_\infty$ , and  $\text{III}(E/K_\infty)(p)$  the  $p$ -primary subgroup of the Tate-Shafarevich of  $E$  over  $K_\infty$ . Let us then write  $X(E/K_\infty) = \text{Hom}(\text{Sel}(E/K_\infty), \mathbf{Q}_p/\mathbf{Z}_p)$  denote the Pontryagin dual of  $\text{Sel}(E/K_\infty)$ , which has the structure of a compact  $\mathcal{O}[[G]]$ -module. Assume that  $E$  has good ordinary reduction at  $p$ . It is then well known that  $X(E/K_\infty)$  has the structure of a torsion  $\mathcal{O}[[G]]$ -module, as shown for instance in [18, Theorem 3.8]. Hence, by the well known structure theory of finitely generated torsion  $\mathcal{O}[[G]]$ -modules, the dual Selmer group  $X(E/K_\infty)$  has an associated  $\mathcal{O}[[G]]$ -characteristic power series  $\text{char } X(E/K_\infty)$ . The two-variable main conjecture of Iwasawa theory then asserts that we have an equality of principal ideals

$$(\mathcal{L}_p(f/K_\infty)) = (\text{char } X(E/K_\infty))$$

in  $\mathcal{O}[[G]]$ . Now, the main result of Skinner-Urban [13] establishes in many cases the divisibility  $(\mathcal{L}_p(f/K_\infty)) \subseteq (\text{char } X(E/K_\infty))$  in  $\mathcal{O}[[G]]$ . The result of Theorem 2.7 in the generic setting with  $k = 0$  can then be viewed as a nontriviality condition for specializations to finite order characters of  $G$  of the  $p$ -adic  $L$ -function  $\mathcal{L}_p(f/K_\infty)$ . Combining these two results allows us to bound the  $\mathcal{O}[[G]]$ -corank of  $\text{Sel}(E/K_\infty)$ , from which we can then obtain bounds for the Mordell-Weil rank of  $E(K_\infty)$  via the exactness of (3). More precisely, we can establish the following types of bounds by this deduction. Recall that we write  $K[p^\infty] = \bigcup_{n \geq 0} K[p^n]$  to denote the union of all ring class fields of  $p$ -power conductor over  $K$ . Recall as well that we write  $\omega$  to denote the quadratic character associated to  $K$ , and moreover that the root number  $\epsilon(1/2, f \times \rho)$  for  $\rho$  any dihedral or ring class character in the setup described above is given by the value  $-\omega(N)$ , at least when the  $N$ ,  $D$  and  $p$  are mutually coprime.

**Theorem 1.4.** *Let  $E/\mathbf{Q}$  be an elliptic curve of conductor  $N$  without complex multiplication, and having good ordinary reduction at  $p$ . Assume for simplicity that  $p \geq 11$ , and that  $(Np, D) = 1$ . Given  $M$  any finite extension of  $K$  contained in  $R_\infty$ , let us write  $r_E(M) = \text{rk}_{\mathbf{Z}} E(M)$  to denote the rank of the Mordell-Weil group  $E(M)$ , i.e. so that  $E(M) \approx \mathbf{Z}^{r_E(M)} \oplus E(M)_{\text{tors}}$ . Then, for any integer  $n \geq 0$ , we have in the ring class tower  $K[p^\infty] = \bigcup_{n \geq 0} K[p^n]$  that*

$$(4) \quad r_E(K[p^n]) = \begin{cases} O_{f,D,p}(1) & \text{if } -\omega(N) = +1 \\ [K[p^n] : K] + O_{f,D,p}(1) & \text{if } -\omega(N) = -1. \end{cases}$$

Moreover, let  $K^{\text{cyc}}$  denote the cyclotomic  $\mathbf{Z}_p$ -extension of  $K$ . Assume that we are not in the exceptional setting, i.e. that for any finite extension  $M$  of  $K$  in the compositum  $K[p^\infty]K^{\text{cyc}}$ ,  $M$  is not contained in  $K[p^\infty]$  if  $-\omega(N) = -1$ . Assume additionally that  $M$  has either (i) sufficiently large dihedral degree, i.e. the dihedral

intersection  $\Omega \cap \text{Gal}(M/K)$  has sufficiently large order, or (ii) trivial dihedral degree, i.e.  $M$  is contained in the cyclotomic  $\mathbf{Z}_p$ -extension  $K^{\text{cyc}}$ . Then,

$$(5) \quad r_E(M) = O_{f,D,p}(1).$$

Moreover, if the criterion of Conjecture 1.2 is established, then (5) holds for all generic extensions  $M$  of  $K$  in the compositum  $K[p^\infty]K^{\text{cyc}}$ .

Here as throughout, we have used the result of Cornut [3] to address the exceptional setting where the root number  $-\omega(N)$  is  $-1$ . The reader should also note that the estimate (4) has already been established via the nonvanishing theorems of Vatsal [19] and Cornut-Vatsal [4], after suitable known Euler system arguments. The new result obtained here is therefore the generic bound (5).

**Some remarks on the general setting.** Though we have not attempted to make the results stated above effective, it is interesting to note that the number of vanishing twists can be bounded above in terms of the Weierstrass degree(s) of the associated two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p(f/K_\infty)$  introduced in Theorem 2.1 below. In fact, after some suitable review of the  $\mathcal{O}[[G]]$ -module structure theory of the associated dual Selmer groups (cf. [18, §3]), it should be possible to establish at least a partial analogue of the conjecture(s) proposed by Coates-Fukaya-Kato-Sujatha [2, §4] in this setting. We have not pursued the matter here. It is also apparent, as in the prequel work [16], that many of these results carry over to higher weight forms, though the nonvanishing theorems of Cornut-Vatsal [4] for totally real fields do not seem to apply directly. Finally, the results described below can be extended to other more general settings, for instance that of Rankin-Selberg  $L$ -functions associated to Hilbert modular forms in abelian extensions of CM fields, as explained in the sequel work [17].

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## 2. NONVANISHING VIA $p$ -ADIC $L$ -FUNCTIONS

We now give the proofs of Theorem 2.7, using only the existence of an associated  $p$ -adic  $L$ -function, i.e. to reduce the problem to previously established results in the self-dual setting. Let us keep all of the notations and setup above.

**Iwasawa algebras.** Let  $\mathcal{O}$  be a finite extension of the  $p$ -adic integers  $\mathbf{Z}_p$ , sufficiently large to contain the ring of integers of the number fields  $F$  and  $F(\mathcal{W})$ . Recall that we write  $R_\infty$  to denote the maximal, abelian unramified outside of  $p$  extension of  $K$ . Recall as well that we write  $\mathcal{G}$  to denote the Galois group  $\text{Gal}(R_\infty/K)$ , so that  $\mathcal{G} \approx G_0 \times G$ , with  $G_0 \approx \text{Gal}(L/K)$  the finite torsion subgroup of  $\mathcal{G}$ , and  $G$  being isomorphic as a topological group to  $\mathbf{Z}_p^2$ . We consider the  $\mathcal{O}$ -Iwasawa algebra  $\mathcal{O}[[\mathcal{G}]]$  of  $\mathcal{G}$ , which is the completed group ring

$$(6) \quad \mathcal{O}[[\mathcal{G}]] = \varprojlim_{\mathcal{U}} \mathcal{O}[\mathcal{G}/\mathcal{U}].$$

Here, the projective limit (6) runs over all open normal subgroups  $\mathcal{U}$  of  $\mathcal{G}$ . More generally, such a definition of  $\mathcal{O}[[\mathcal{G}]]$  can be made for  $\mathcal{O}$  any discrete valuation ring and  $\mathcal{G}$  any profinite group. The reader should note that the elements of any such completed group ring  $\mathcal{O}[[\mathcal{G}]]$  can be viewed as  $\mathcal{O}$ -valued measures on  $\mathcal{G}$  in a natural way. More precisely, given  $\mathcal{W}$  any finite order character of  $\mathcal{G}$ , and  $\mathcal{L}$  any element of the completed group ring  $\mathcal{O}[[\mathcal{G}]]$ , we can integrate  $\mathcal{W}$  against  $\mathcal{L}$  in the following

way. Since  $\mathcal{W}$  is of finite order, it defines a locally constant function on  $\mathcal{G}$ , and hence there exists an open subgroup  $\mathcal{U} \subset \mathcal{G}$  such that  $\mathcal{W}$  is constant modulo  $\mathcal{U}$ . Writing

$$\mathcal{L}_{\mathcal{U}} = \sum_{\sigma \in \mathcal{G}/\mathcal{U}} c_{\mathcal{U}}(\sigma) \sigma$$

to denote the image of  $\mathcal{L}$  in the group ring  $\mathcal{O}[\mathcal{G}/\mathcal{U}]$ , with coefficients  $c_{\mathcal{U}}(\sigma) \in \mathcal{O}$ , we can then define the integral of  $\mathcal{W}$  against  $\mathcal{L}$  to be the finite sum

$$(7) \quad \int_{\mathcal{G}} \mathcal{W}(\sigma) d\mathcal{L}(\sigma) = \sum_{\sigma \in \mathcal{G}/\mathcal{U}} c_{\mathcal{U}}(\sigma) \mathcal{W}(\sigma).$$

It is easy to see that this definition does not depend on the choice of open subgroup  $\mathcal{U} \subset \mathcal{G}$ . Thus, given an element  $\mathcal{L}$  in  $\mathcal{O}[[\mathcal{G}]]$ , we write  $d\mathcal{L}$  to denote the associated measure, which is determined uniquely by this construction. We shall also write  $\mathcal{W}(\mathcal{L})$  to denote the functional defined in (7) above, and moreover refer to this value as the *specialization of  $\mathcal{L}$  to  $\mathcal{W}$* . It is easy to see from this description that any group like element  $g \in \mathcal{G}$  corresponds to the Dirac measure  $dg$ , and that the product  $\mathcal{L}_1 \mathcal{L}_2$  of two elements  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{O}[[\mathcal{G}]]$  corresponds to the convolution product  $d\mathcal{L}_1 \star d\mathcal{L}_2$ . Moreover, the identity element  $\mathcal{I}$  in  $\mathcal{O}[[\mathcal{G}]]$  corresponds to a constant measure. Now, returning to the specific setting of  $\mathcal{G} = \text{Gal}(R_{\infty}/K)$ , we have a natural identification of completed group rings  $\mathcal{O}[[\mathcal{G}]] \approx \mathcal{O}[G_0][[G]]$ . We also have an isomorphism of completed group rings

$$(8) \quad \mathcal{O}[[\mathcal{G}]] \approx \mathcal{O}[G_0][[G]] \longrightarrow \bigoplus_{\mathcal{W}_0} \mathcal{O}[[G]], \quad \mathcal{L} \longmapsto (\mathcal{W}_0(\mathcal{L}))_{\mathcal{W}_0},$$

where the direct sum runs over all characters  $\mathcal{W}_0$  of the finite group  $G_0$ , and  $(\mathcal{W}_0(\mathcal{L}))_{\mathcal{W}_0}$  is the vector of specializations  $\mathcal{W}_0(\mathcal{L})$  of  $\mathcal{L}$  to each character  $\mathcal{W}_0$ . The reader should note that here, we only specialize to the tamely ramified part  $\mathcal{W}_0$  (and not to any wildly ramified character of  $\mathcal{G}$ ), so that the  $\mathcal{W}_0(\mathcal{L})$  are elements of the completed group ring  $\mathcal{O}[[G]]$  rather than just values in the ring of integers  $\mathcal{O}$ . To denote this distinction more clearly, we shall write  $\mathcal{L}(\mathcal{W}_0)$  rather than  $\mathcal{W}_0(\mathcal{L})$  is what follows to emphasize that each  $\mathcal{L}(\mathcal{W}_0)$  is a genuine element of  $\mathcal{O}[[G]]$ .

**Two-variable  $p$ -adic  $L$ -functions.** The constructions of Hida [6] and Perrin-Riou [8] give us the following result. Recall that by the theorem of Shimura [11], the values

$$(9) \quad \frac{L(1/2, f \times \mathcal{W})}{8\pi^2 \langle f, f \rangle}$$

are algebraic, and in fact contained in  $F(\mathcal{W})$ . Let us now fix an embedding  $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ , where  $\overline{\mathbf{Q}}_p$  is a fixed algebraic closure of  $\mathbf{Q}_p$ . We can then view any element of  $\overline{\mathbf{Q}}$  as an element of  $\overline{\mathbf{Q}}_p$ . In particular, we shall view the values (9) as elements of  $\overline{\mathbf{Q}}_p$ .

**Theorem 2.1** (Hida, Perrin-Riou). *Assume that  $f$  is  $p$ -ordinary, and that  $p \geq 5$ . There exists an element  $\mathcal{L}_p = \mathcal{L}_p(R_{\infty}/K)$  in the  $\mathcal{O}$ -Iwasawa algebra  $\mathcal{O}[[\mathcal{G}]]$  such that for any finite order character  $\mathcal{W}$  of  $\mathcal{G}$ , we have the equality*

$$(10) \quad \mathcal{W}(\mathcal{L}_p) = \eta(f, \mathcal{W}) \cdot \frac{L(1/2, f \times \overline{\mathcal{W}})}{8\pi \langle f, f \rangle} \in \overline{\mathbf{Q}}_p.$$



Here,  $\eta(f, \mathcal{W})$  denotes some precise, nonvanishing algebraic number, viewed as an element of  $\overline{\mathbf{Q}}_p$ . In particular, the central value  $L(1/2, f \times \overline{\mathcal{W}})$  vanishes if and only if the specialization  $\mathcal{W}(\mathcal{L}_p)$  vanishes.

*Proof.* The result follows from Perrin-Riou [8, Théorème B], using the bounded linear form construction of Hida [6], as explained in [18, Theorem 2.9]. That is, the integrality of this construction is explained in [18, Theorem 2.9], assuming for simplicity that  $p \geq 5$ . Note that this construction, which also works for higher weight forms, requires that the eigenform  $f$  be  $p$ -ordinary.  $\square$

We shall say that the  $p$ -adic  $L$ -function  $\mathcal{L}_p = \mathcal{L}_p(f/R_\infty)$  in  $\mathcal{O}[[\mathcal{G}]]$  interpolates the central values  $L(1/2, f \times \overline{\mathcal{W}})$  once such a formula (10) is known. More generally, given  $\mathcal{O}$  any discrete valuation ring and  $\mathcal{G}$  any profinite group, we shall say that an element  $\mathcal{L}$  in  $\mathcal{O}[[\mathcal{G}]]$  interpolates some complex value  $\eta$  if the specialization  $\mathcal{W}(\mathcal{L}_p)$  equals  $\eta/\vartheta$  in  $\overline{\mathbf{Q}}_p$  for  $\mathcal{W}$  some finite order character of  $\mathcal{G}$  and  $\vartheta = \vartheta_\eta$  some suitable period (i.e. for which the quotient  $\eta/\vartheta$  lies in  $\overline{\mathbf{Q}}$ ).

**Weierstrass preparation.** We now review the Weierstrass preparation theorem, and in particular how it applies to elements of the completed group ring  $\mathcal{O}[[G]]$ . Let us fix topological generators  $\gamma_1$  and  $\gamma_2$  of the Galois group  $G = \Omega \times \Gamma$ , where  $\gamma_1$  is a topological generator of the anticyclotomic factor  $\Omega \cong \mathbf{Z}_p$ , and  $\gamma_2$  is a topological generator of the cyclotomic factor  $\Gamma \cong \mathbf{Z}_p$ . We may then invoke the well known isomorphism of completed group rings

$$(11) \quad \mathcal{O}[[G]] \longrightarrow \mathcal{O}[[T_1, T_2]], \quad (\gamma_1, \gamma_2) \longmapsto (T_1 + 1, T_2 + 1)$$

to view each of the specialized  $p$ -adic  $L$ -functions  $\mathcal{L}_p(\mathcal{W}_0)$  in (8) above as an element  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  of the two-variable power series ring  $\mathcal{O}[[T_1, T_2]]$ . The reader should note that, under this non-canonical isomorphism (11), the specialization  $\mathcal{W}(\mathcal{L}_p) = \mathcal{W}_w(\mathcal{L}_p(\mathcal{W}_0)) = \rho_w \chi \circ \mathbf{N}(\mathcal{L}_p(\mathcal{W}_0))$  corresponds to evaluating the power series  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2) = \mathcal{L}_p(\mathcal{W}_0; \gamma_1 - 1, \gamma_2 - 1)$  at a certain pair of primitive roots of unity  $\zeta_1$  and  $\zeta_2$ , so that

$$\mathcal{W}(\mathcal{L}_p(T_1, T_2)) = \mathcal{L}_p(\mathcal{W}_0; \rho_w(\gamma_1) - 1, \psi_w(\gamma_2) - 1) = \mathcal{L}_p(\mathcal{W}_0; \zeta_1 - 1, \zeta_2 - 1).$$

That is, these roots of unity  $\zeta_1$  and  $\zeta_2$  are determined uniquely by the values  $\rho_w(\gamma_1) = \zeta_1$  and  $\psi_w(\gamma_2) = \zeta_2$ , where  $\rho_w$  denotes the wildly ramified part of the ring class character  $\rho$ , and  $\psi_w = \chi \circ \mathbf{N}$  denotes the (wildly ramified) cyclotomic part of  $\mathcal{W} = \rho \chi \circ \mathbf{N}$ .

Let us now write  $R_i[[T_j]]$  to denote any of the one-variable power series rings  $\mathcal{O}[[T]]$ ,  $\mathcal{O}[[T_1]][[T_2]]$  or  $\mathcal{O}[[T_2]][[T_1]]$ , i.e. so that  $R_i = \mathcal{O}$  or else  $R_i = \mathcal{O}[[T_i]]$  with  $i, j \in \{1, 2\}$  and  $i \neq j$ . Since  $\mathcal{O}$  is a local ring, it has a unique maximal ideal  $\mathfrak{P}$  say. Now, it is well known and easy to show that each  $\mathcal{O}[[T_i]]$  is a local ring, with unique maximal ideal  $(\mathfrak{P}, T_i)$ . Thus, each choice of  $R_i$  is a local ring with maximal ideal  $\mathfrak{m}_i$  say. An element  $f(T_j)$  of the group ring  $R_i[T_j]$  is said to be a *distinguished* (or *Weierstrass*) *polynomial* if it takes the form

$$f(T_j) = T_j^r + b_{r-1}(T_i)T_j^{r-1} + \dots + b_0(T_i),$$

where each of the coefficients  $b_{r-1}(T_i), \dots, b_0(T_i)$  lies in the maximal ideal  $\mathfrak{m}_i$ . Suppose now that we have an element  $g(T_j)$  of the formal power series ring  $R_i[[T_j]]$ ,

$$g(T_j) = \sum_{k \geq 0} a_k(T_i)T_j^k,$$

such that not all of the coefficients  $a_k(T_j) \in R_i$  lie in the maximal ideal  $\mathfrak{m}_i$ . Say  $a_0(T_i), \dots, a_{r-1}(T_i) \in \mathfrak{m}_i$  for some integer  $r \geq 1$ , with  $a_r(T_i)$  a unit in  $R_i$ . The Weierstrass preparation theorem asserts that this  $g(T_j)$  can be written uniquely as

$$g(T_j) = f_i(T_j)u_i(T_j),$$

where  $f_i(T_j)$  is a distinguished polynomial in  $R_i[[T_j]]$  of degree  $r$ , and  $u_i(T_j)$  is a unit in  $R_i[[T_j]]$ . The integer  $r \geq 1$  is then known as the *Weierstrass degree*  $\deg_W(g(T_j))$  of  $g(T_j)$ . More generally, if

$$g(T_j) = \sum_{k \geq 0} a_k(T_i)T_j^k,$$

is a nonzero power series in  $R_i[[T_j]]$ , then the Weierstrass preparation theorem asserts that  $g(T_j)$  can be expressed uniquely as a product

$$(12) \quad g(T_j) = f_i(T_j)u_i(T_j)\varpi_i(T_i),$$

where  $f_i(T_j)$  and  $u_i(T_j)$  are as above, and  $\varpi_i(T_i)$  is the element of  $R_i = \mathcal{O}[[T_i]]$  determined by taking the greatest common divisor of all of the coefficients  $a_k(T_i)$  contained in the maximal ideal  $\mathfrak{m}_i = (\mathfrak{P}, T_i)$ . Now, as the invertible power series  $u(T_j)$  cannot have any zeros, it follows that any nonzero element  $g(T_j)$  of  $R_i[[T_j]]$  has at most  $\deg_W(g(T_j))$  zeros in the indeterminate  $T_j$ . This result can be interpreted for the  $p$ -adic  $L$ -functions  $\mathcal{L}_p(\mathcal{W}_0)$  on the level of specializations to wildly ramified characters in either the anticyclotomic variable  $T_1 = \gamma_1 - 1$  or the cyclotomic variable  $T_2 = \gamma_2 - 1$ , as we shall see more precisely below. Let us for now just record the following observation about each of the two-variables  $p$ -adic  $L$ -functions  $\mathcal{W}_0(\mathcal{L}_p)$  in  $\mathcal{O}[[G]]$ , viewed as power series  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  in  $\mathcal{O}[[T_1, T_2]]$  under the fixed isomorphism (11). As explained below, we may assume that each such  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  is not identically zero. Viewing  $\mathcal{L}_p(\rho_0; T_1, T_2)$  as an element of the one-variable power series ring  $R_i[[T_j]] = \mathcal{O}[[T_i]][[T_j]]$ , the Weierstrass preparation theorem then implies that  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  can be expressed uniquely as a product of the form (12) above. Let us write  $r(j)$  to denote the Weierstrass degree of  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  in  $R_i[[T_j]]$ . Observe that we may also apply the Weierstrass preparation theorem to the element  $\varpi_i(T_i)$  of  $R_i$ . Let us then write  $\deg_W(\varpi_i)$  to denote its Weierstrass degree. Putting together the two unique expressions (12) for  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$ , it is then easy to see from the induced relations that

$$(13) \quad r(1) \deg_W(\varpi_2) = r(2) \deg_W(\varpi_1).$$

It is also easy to see that  $\deg_W(\varpi_2)$  is bounded above by the Weierstrass degree of the cyclotomic  $p$ -adic  $L$ -function  $\mathcal{L}_p(\mathcal{W}_0; 0, T_2)$  in  $R_2$ , and that  $\deg_W(\varpi_1)$  is bounded above by the Weierstrass degree of the anticyclotomic  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1, 0)$  in  $R_1$ . The arguments below will show that both sides of (13) are finite, i.e. defined.

**Nontriviality of tamely ramified specializations (via basechange).** We begin with the following result, deduced via basechange from the nonvanishing theorems of Cornut-Vatsal [4] for totally real fields. This result applies to the vector of  $p$ -adic  $L$ -functions  $(\mathcal{L}_p(\mathcal{W}_0))_{\mathcal{W}_0} = (\mathcal{L}_p(\mathcal{W}_0; T_1, T_2))_{\mathcal{W}_0}$  appearing in (8) above, in particular to show that each  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  is nontrivial in the sense that its specialization to infinitely many wildly ramified characters  $\mathcal{W}_w = \rho_w \psi_w$  (with both  $\rho_w$  and  $\psi_w$  nontrivial) does not vanish. Thus, by the Weierstrass preparation theorem,

this result will imply that each  $\mathcal{L}_p(\mathcal{W}_0; T_1, T_2)$  has finite Weierstrass degree as an element in either of the power series rings  $\mathcal{O}[[T_1]][[T_2]]$  or  $\mathcal{O}[[T_2]][[T_1]]$ .

Let us first fix a character  $\mathcal{W}_0$  of  $G_0$ , which recall is a tamely ramified character of some  $p$ -power conductor. Observe that we can always take such a  $\mathcal{W}_0$  to be a ring class character  $\rho_0$ , as there are no unramified cyclotomic characters of  $p$ -power conductor apart from the trivial one. Let us now consider the following basechange setup. Fix  $n \geq 0$  an integer. Let  $\zeta_{p^n}$  be a primitive  $p^n$ -th root of unity, with  $\mathbf{Q}(\zeta_{p^n})$  the field obtained by adjoining  $\zeta_{p^n}$  to  $\mathbf{Q}$ . We then write  $\mathbf{Q}_n = \mathbf{Q}(\zeta_{p^{n+1}})^+$  to denote the maximal totally real subfield of  $\mathbf{Q}(\zeta_{p^{n+1}})$ . Hence,  $\mathbf{Q}_n$  is the unique extension of degree  $p^n$  over  $\mathbf{Q}$  in the cyclotomic  $\mathbf{Z}_p$ -extension of  $\mathbf{Q}$ . Let us also write  $K_n$  to denote the compositum  $K\mathbf{Q}_n$ . Hence,  $K_n$  is the unique extension of degree  $p^n$  over  $K$  in the cyclotomic  $\mathbf{Z}_p$ -extension of  $K$ . Clearly,  $K_n$  is a totally imaginary quadratic extension of the totally real field  $\mathbf{Q}_n$ . Moreover, the cyclotomic field  $\mathbf{Q}_n$  is abelian, and of odd degree. Thus, writing  $\pi_f$  to denote the cuspidal automorphic representation of  $\mathrm{GL}_2(\mathbf{Q})$  associated to  $f$ , there exists by the theory of cyclic basechange a cuspidal automorphic representation  $\Pi_{f,n}$  of  $\mathrm{GL}_2(\mathbf{Q}_n)$  such that there is an equality of  $L$ -functions

$$L(s, \Pi_{f,n}) = \prod_{\chi} L(s, \pi_f, \chi).$$

Here,  $L(s, \Pi_{f,n})$  denotes the  $L$ -function of  $\Pi_{f,n}$ , the product runs over all characters  $\chi$  of  $\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})$ , and each  $L(s, \pi_f, \chi)$  denotes the  $L$ -function of  $\pi_f$  twisted by  $\chi$ . Now, since  $\mathbf{Q}_n$  is totally ramified at  $p$ , there exists a unique prime  $\mathfrak{p}$  above  $p$  in  $\mathbf{Q}_n$ . Following Cornut-Vatsal [4, § 1.2], we then write  $K_n[\mathfrak{p}^\infty] = \bigcup_{m \geq 0} K_n[\mathfrak{p}^m]$  to denote the tower of all ring class extensions of  $\mathfrak{p}$ -power conductor over  $K_n$ , with  $\Omega^{(n)}$  to denote its Galois group  $\mathrm{Gal}(K_n[\mathfrak{p}^\infty]/K_n)$ . Let us now view any finite order character  $\rho^{(n)}$  of  $\Omega^{(n)}$  as an idele class character of  $K_n$  via the reciprocity map of class field theory. We can then consider for any such character the Rankin-Selberg  $L$ -function  $L(s, \Pi_{f,n} \times \rho^{(n)})$  of  $\Pi_{f,n}$  times the automorphic representation of  $\mathrm{GL}_2(\mathbf{Q}_n)$  associated to  $\rho^{(n)}$ . This  $L$ -function has a well known analytic continuation, and satisfies a functional equation relating values at  $s$  to  $1-s$ . Moreover, it is self dual, its corresponding root number  $\epsilon(1/2, \Pi_{f,n} \times \rho^{(n)})$  taking values in  $\pm 1$ , and can be described by a well known formula (see for instance [4, § 1.1]), completely analogously to the setting with  $\epsilon(1/2, f \times \rho) = -\omega(N)$  described above. On the other hand, given  $\rho$  any finite order ring class character of the imaginary quadratic field  $K$  factoring through the Galois group  $\mathcal{G}$ , let us write  $\rho'$  to denote the composition of  $\rho$  with the norm homomorphism  $\mathbf{N}_{K_n/K}$  from  $K_n$  to  $K$ , so that  $\rho'$  defines a finite order character of the Galois group  $\Omega^{(n)}$ . We then have for any such character  $\rho'$  an equality of  $L$ -functions

$$(14) \quad L(s, \Pi_{f,n} \times \rho') = \prod_{\chi} L(s, f \times \rho\chi \circ \mathbf{N}),$$

where the product runs over all characters  $\chi$  of  $\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})$  as before. This equality of  $L$ -functions induces an equality of the associated root numbers

$$(15) \quad \epsilon(1/2, \Pi_{f,n} \times \rho') = \prod_{\chi} \epsilon(1/2, f \times \rho\chi \circ \mathbf{N}).$$

In particular, using that the degree of  $\mathbf{Q}_n$  is always odd by our hypothesis that  $p \geq 5$ , we can deduce from this relation (15) that the condition of having our base

root number  $\epsilon(1/2, f \times \mathbf{1}_K) = -\omega(N)$  equal  $\pm 1$  will imply that the root number  $\epsilon(1/2, \Pi_{f,n} \times \rho')$  equals  $\pm 1$  for any such character  $\rho'$  of  $\Omega^{(n)}$ . This puts us in a good position to invoke the nonvanishing theorems of Cornut-Vatsal [4] directly in either case on the root number  $\epsilon(1/2, f \times \mathbf{1}_K)$ . More precisely, we obtain the following version of their result(s) in this setting after invoking the Artin formalism of (14) and (15) above.

**Proposition 2.2.** *Fix a tamely ramified character  $\mathcal{W}_0 = \rho_0$  of  $G_0$ . Let  $n \geq 0$  be any integer. There exists a positive integer  $c(n) \geq 0$  such that for all ring class conductors  $c \geq c(n)$ , the associated Galois average  $\delta_{c,p^n;\rho_0}^{(k)}$  does not vanish. Here,  $k = 0$  or  $1$  according as to whether or not the pair  $(f, \mathcal{W})$  is generic or exceptional.*

*Proof.* We can assume without loss of generality that  $n \geq 1$ , since the case of  $n = 0$  is shown already by the nonvanishing theorems of [3], [4] and [19]. Thus, let us fix an integer  $n \geq 1$ . In particular, this means that we need only work in the generic setting with  $k = 0$ . We now divide into cases on the root number  $\epsilon(1/2, f \times \mathbf{1}_K)$ .

Let us first suppose that the root number  $\epsilon(1/2, f \times \mathbf{1}_K) = -\omega(N)$  is  $+1$ . We can then deduce from Cornut-Vatsal [4, Theorem 1.4] that for all but finitely many finitely order characters  $\rho^{(n)}$  of  $\Omega^{(n)}$ , the value  $L(1/2, \Pi_{f,n} \times \rho_n)$  does not vanish. The reader should note that this is not strictly what is stated in [4, Theorem 1.4], but rather what can be deduced from algebraicity (using for instance the main theorem of Shimura [12] for Hilbert modular forms, or the existence of an associated  $p$ -adic  $L$ -function as given in [15]). We can then deduce that for all ring class characters  $\rho = \rho_0 \rho_w$  of  $K$  factoring through  $\mathcal{G}$  of sufficiently large conductor that the value

$$L(1/2, \Pi_{f,n} \times \rho') = \prod_{\chi} L(1/2, f \times \rho \chi \circ \mathbf{N})$$

does not vanish. Here, the product again runs over all characters  $\chi$  of  $\text{Gal}(\mathbf{Q}_n/\mathbf{Q})$ , and sufficiently large conductor means that  $\rho$  has conductor  $c \geq c_0(n)$ , where  $c_0(n) = c_0(n, f, p, K)$  is some positive integer depending on the choice of cyclotomic field  $\mathbf{Q}_n$ . Thus, for each integer  $n \geq 0$ , we can find some ring class character  $\rho = \rho_0 \rho_w$  of conductor  $c \geq c_0(n)$  such that the central value  $L(1/2, f \times \rho \chi \circ \mathbf{N})$  does not vanish for any Dirichlet character  $\chi$  of  $\text{Gal}(\mathbf{Q}_n/\mathbf{Q})$ . Now, fixing a character  $\chi$  of  $\text{Gal}(\mathbf{Q}_n/\mathbf{Q})$ , we can deduce from the algebraicity theorem of Shimura [12] that the value  $L(1/2, f \times \rho \chi \circ \mathbf{N})$  does not vanish for all primitive ring class characters  $\rho = \rho_0 \rho_w$  of conductor  $c$ . Moreover, for any fixed primitive ring class character  $\rho = \rho_0 \rho_w$  of conductor  $c$ , we can deduce from another application of the algebraicity theorem of Shimura [12] that the value  $L(1/2, f \times \rho \chi \circ \mathbf{N})$  does not vanish for all primitive Dirichlet characters  $\chi$  of conductor  $p^n$ . Thus, we can deduce from algebraicity that the value  $L(1/2, f \times \rho \chi \circ \mathbf{N})$  does not vanish for any primitive character  $\mathcal{W} = \rho \chi \circ \mathbf{N} \in P_{c,p^n;\rho_0}$ . Equivalently, the Galois average  $\delta_{c,p^n;\rho_0}^{(0)}$  does not vanish for all ring class conductors  $c \geq c_0(n)$ .

Let us now suppose that the root number  $\epsilon(1/2, f \times \mathbf{1}_K)$  is  $-1$ . We can then deduce from Cornut-Vatsal [4, Theorem 1.5], that for all but finitely many characters  $\rho^{(n)}$  of  $\Omega^{(n)}$ , the value  $L'(1/2, \Pi_{f,n} \times \rho_n)$  does not vanish. The reader should note again that this is not strictly what is stated in [4, Theorem 1.5], but rather what can be deduced from algebraicity using special value formulae (in this case, the recent work of Yuan-Zhang-Zhang [21]). Anyhow, we then have that for all but finitely many characters  $\rho^{(n)}$  of  $\Omega^{(n)}$ ,  $\text{ord}_{s=1/2} L(s, \Pi_{f,n} \times \rho^{(n)}) = 1$ . It follows

that for ring class character  $\rho = \rho_0 \rho_w$  of  $K$  factoring through  $\mathcal{G}$  of sufficiently large conductor, the value  $L'(1/2, \Pi_{f,n} \times \rho')$  does not vanish, whence

$$(16) \quad \text{ord}_{s=1/2} L(s, \Pi_{f,n} \times \rho') = \sum_{\chi} \text{ord}_{s=1/2} L(s, f \times \rho \chi \circ \mathbf{N}) = 1.$$

by the decomposition (14). Here, sufficiently large conductor means that  $\rho$  has conductor  $c \geq c_1(n)$ , where  $c_1(n) = c_1(n, f, p, K)$  is some positive integer depending on the choice of cyclotomic field  $\mathbf{Q}_n$ . Now, since the result of [4, Theorem 1.5] also applies in the same way to the base field  $\mathbf{Q}_0 = \mathbf{Q}$ , we can assume without loss of generality that  $\text{ord}_{s=1/2} L(s, f \times \rho) = 1$ , whence (16) implies that

$$\sum_{\chi \neq 1} \text{ord}_{s=1/2} L(s, f \times \rho \chi \circ \mathbf{N}) = 0.$$

Thus, using the same reasoning as before, we can deduce that the Galois average  $\delta_{c,p^n;\rho_0}^{(0)}$  does not vanish for all ring class conductors  $c \geq c_1(n)$ .  $\square$

**Corollary 2.3.** *Let  $c_0 = \min_n c(n)$ . Then, for each ring class conductor  $c \geq c_0$ , there exists an integer  $m(c) \geq 0$  such that for all integers  $n \geq m(c)$ , the associated Galois average  $\delta_{c,p^n;\rho_0}^{(k)}$  does not vanish. Here,  $k = 0$  or  $1$  according as to whether or not the pair  $(f, \mathcal{W})$  is generic or exceptional.*

*Proof.* Recall that we fix a tamely ramified character  $\mathcal{W}_0 = \rho_0$ . Consider the associated  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1, T_2)$  in  $\mathcal{O}[[T_1, T_2]]$ . Fix a ring class of  $(p$ -power) conductor  $c \geq c_0$ . Let  $\rho = \rho_0 \rho_w$  be a primitive ring class character of conductor  $c$ . Consider the (doubly) specialized  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; \rho_w(\gamma_1) - 1, T_2)$ , in  $\mathcal{O}[[T_2]]$ . We know by the Weierstrass preparation theorem that this element can be expressed uniquely as a product  $u(T_2)f(T_2)\varpi^\mu$ , where  $u(T_2)$  is an invertible power series in  $\mathcal{O}[[T_2]]$ ,  $f(T_2)$  is a distinguished polynomial in  $\mathcal{O}[T_2]$ ,  $\varpi$  is a uniformizer for  $\mathcal{O}$ , and  $\mu \geq 0$  is an integer. We also know by Proposition 2.2 that for some integer  $h \geq 0$ , the associated Galois average  $\delta_{c,p^h;\rho_0}^{(0)}$  does not vanish. Hence, we can deduce from the interpolation property of Theorem 2.1 that the specialized  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; \rho_w(\gamma_1) - 1, T_2)$  has finite Weierstrass degree  $\deg_W(\mathcal{L}_p(\rho_0; \rho_w(\gamma_1) - 1, T_2))$  as an element of  $\mathcal{O}[[T_2]]$ . Thus,  $\mathcal{L}_p(\rho_0; \rho_w(\gamma_1) - 1, T_2)$  can have at most  $\deg_W(\mathcal{L}_p(\rho_0; \rho_w(\gamma_1) - 1, T_2))$  many zeros. It then follows from Shimura's algebraicity theorem [12] that  $\delta_{c,p^h;\rho_0}^{(0)}$  does not vanish for all but finitely many integers  $h \geq 0$ . Writing  $l = l(c) \geq 0$  to denote the largest integer for which  $\delta_{c,p^l;\rho_0}^{(0)}$  vanishes, it follows that  $\delta_{c,p^n;\rho_0}^{(0)}$  does not vanish for all integers  $n \geq l(c) + 1$ . Thus, taking  $m(c) = l(c) + 1$  proves the claim for  $c$ .  $\square$

**Basechange  $p$ -adic  $L$ -functions.** Let us keep all of the notations of the paragraph above. Hence, we fix an integer  $n \geq 0$ , writing  $\zeta_{p^n}$  to denote a primitive  $p^n$ -th power root of unity. We then write  $\mathbf{Q}(\zeta_{p^n})$  to denote the field obtained by adjoining  $\zeta_{p^n}$  to  $\mathbf{Q}$ , with  $\mathbf{Q}_n = \mathbf{Q}(\zeta_{p^{n+1}})^+$  the maximal totally real subfield of  $\mathbf{Q}(\zeta_{p^{n+1}})$ . Thus,  $\mathbf{Q}_n$  is the degree- $p^n$  extension of  $\mathbf{Q}$  contained in the cyclotomic  $\mathbf{Z}_p$ -extension of  $\mathbf{Q}$ . Writing  $K_n$  to denote the compositum  $K\mathbf{Q}_n$ , we see that  $K_n$  is the degree- $p^n$  extension of  $K$  contained in the cyclotomic  $\mathbf{Z}_p$ -extension of  $K$ , and moreover that  $K_n$  is a totally imaginary quadratic over  $\mathbf{Q}_n$ . Recall as well that we write  $\mathfrak{p}$  to denote the unique prime above  $p$  in  $\mathbf{Q}_n$ , with  $K_n[\mathfrak{p}^\infty] = \bigcup_{m \geq 0} K_n[\mathfrak{p}^m]$  the  $\mathfrak{p}^\infty$ -ring class tower of  $K_n$ , and  $\Omega^{(n)} = \text{Gal}(K_n[\mathfrak{p}^\infty]/K_n)$  its Galois group. On

the other hand, recall that we write  $D_\infty = K_{p^\infty}$  to denote the dihedral or anti-cyclotomic  $\mathbf{Z}_p$ -extension of the imaginary quadratic field  $K$ , so that we have an identification of  $\Omega$  with the Galois group  $\text{Gal}(D_\infty/K)$ . Let us then write  $\Omega^{(n)}$  to denote the Galois group  $\text{Gal}(K_n D_\infty/K_n)$ , where  $K_n D_\infty$  denotes the compositum of the finite cyclotomic extension  $K_n$  with  $D_\infty$ . Hence,  $\Omega^{(n)}$  is topologically isomorphic to  $\mathbf{Z}_p$ . Now, recall that we fixed a topological generator  $\gamma_1$  of  $\Omega = \Omega^{(0)}$ . Let us fix a topological generator  $\gamma_1^{(n)}$  of  $\Omega^{(n)}$  lifting  $\gamma_1$  that is compatible with respect to composition with the norm homomorphism  $\mathbf{N}_{K_n/K}$  on the associated character groups. We can then invoke the standard isomorphism of completed group rings

$$(17) \quad \mathcal{O}[[\Omega^{(n)}]] \longrightarrow \mathcal{O}[[T_1^{(n)}]], \quad \gamma_1^{(n)} \longmapsto T_1^{(n)} + 1.$$

Moreover, we can define from each two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1, T_2)$  described above a basechange  $p$ -adic  $L$ -function in the lifted variable  $T_1^{(n)}$ ,

$$(18) \quad \mathcal{L}_p(\rho_0; T_1^{(n)}) = \prod_{\psi_w} \mathcal{L}_p(\rho_0; T_1, \psi_w(\gamma_2) - 1) \in \mathcal{O}[[T_1^{(n)}]].$$

Here, the product ranges over all (wildly ramified) characters  $\psi_w = \chi \circ \mathbf{N}$  of the Galois group  $\text{Gal}(K_n/K)$ . Equivalently, by definition of the norm homomorphism  $\mathbf{N}_{K_n/K}$ , we have the defining relation

$$(19) \quad \mathcal{L}_p(\rho_0; T_1^{(n)}) = \mathcal{L}_p(\rho_0; T_1, T_2) \circ \mathbf{N}_{K_n/K} \in \mathcal{O}[[T_1^{(n)}]],$$

where the composition with the norm homomorphism  $\mathbf{N}_{K_n/K}$  is taken on the level of the associated completed group ring element in  $\mathcal{O}[[G]] \cong \mathcal{O}[[\Omega]][[\Gamma]]$ . On the other hand, after taking images of these group ring element(s) under (17), we claim it is a formal consequence of the definition of the topological generator  $\gamma_1^{(n)}$  that the Weierstrass degree of  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  in  $\mathcal{O}[[T_1^{(n)}]]$  must equal the Weierstrass degree of  $\mathcal{L}_p(\rho_0; T_1^{(0)}) = \mathcal{L}_p(\rho_0; T_1, 0)$  in  $\mathcal{O}[[T_1^{(0)}]] = \mathcal{O}[[T_1]]$ . Note that we have adopted the convention of taking the Weierstrass degree to be infinite in the event that  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  is identically zero in  $\mathcal{O}[[T_1^{(n)}]]$ . Let us now record these observations (with some additional justification) as follows.

**Lemma 2.4.** *Fix a tamely ramified character  $\mathcal{W}_0 = \rho_0$ . There exists a positive integer  $n(0)$  such that the following property holds. For each integer  $n \geq n(0)$ , the Weierstrass degree of  $\mathcal{L}(\rho_0; T_1^{(n)})$  as an element of  $\mathcal{O}[[T_1^{(n)}]]$  equals the Weierstrass degree of  $\mathcal{L}_p(\rho_0; T_1^{(0)}) = \mathcal{L}_p(\rho_0; T_1, 0)$  as an element of  $\mathcal{O}[[T_1^{(0)}]] = \mathcal{O}[[T_1]]$ .*

*Proof.* We know by the Weierstrass preparation theorem that  $\mathcal{L}(\rho_0; T_1, T_2)$  as an element of  $R_2[[T_1]] = \mathcal{O}[[T_2]][[T_1]]$  can be expressed uniquely as

$$\mathcal{L}_p(\rho_0; T_1, T_2) = u_2(T_1) f_2(T_1) \varpi_2(T_2).$$

Here,  $u_2(T_1)$  is unit in  $R_2[[T_1]]$ ,  $f_2(T_1)$  is a distinguished polynomial in  $R_2[[T_1]]$ , and  $\varpi_2(T_2)$  is a power series in  $\mathfrak{m}_2 \subset R_2$  as in (12) above. Thus, we can write

$$f_2(T_1) = T_1^r + b_{r-1}(T_2) T_1^{r-1} + \dots + b_0(T_2),$$

for  $r \geq 0$  an integer, with each  $b_i(T_2)$  contained in the maximal ideal  $\mathfrak{m}_2 = (\mathfrak{P}, T_2)$ . Recall that we write  $\deg_W(\varpi_2)$  to denote the Weierstrass degree of  $\varpi_2(T_2)$  in  $R_2$ . Observe that the specialization  $\varpi_2(\psi_w(\gamma_2) - 1)$  can then only vanish for at most finitely many cyclotomic characters  $\psi_w$  of  $\Gamma$ , i.e. for  $\deg_W(\varpi_2)$  many characters. In particular, we can deduce that there exists a (minimal) positive integer  $q(0)$  such

that for all cyclotomic character  $\psi_w$  of ( $p$ -power) conductor  $q \geq q(0)$ , the specialization  $\varpi_2(\psi_w(\gamma_2) - 1)$  does not vanish. In particular, specializing  $\mathcal{L}_p(\rho_0; T_1, T_2)$  to any such character  $\psi_w$  of  $\Gamma$  does not change the Weierstrass degree in  $R_2[[T_1]]$ . That is, we have in this case that

$$(20) \quad \deg_W(\mathcal{L}_p(\rho_0; T_1, T_2)) = \deg_W(\mathcal{L}_p(\rho_0; T_1, \psi_w(\gamma_2) - 1))$$

as elements of  $R_2[[T_1]]$  for any such character  $\psi_w$ . This is a consequence of the fact that the Weierstrass degree  $\deg_W(\mathcal{L}_p(\rho_0; T_1, \psi_w(\gamma_2) - 1))$  of  $\mathcal{L}_p(\rho_0; T_1, \psi_w(\gamma_2) - 1)$  in  $\mathcal{O}[[T_1]] \subset R_2[[T_1]]$  is given by that of the specialized Weierstrass polynomial

$$f_2(T_1)|_{\psi_w} := T_1^r + b_{r-1}(\psi_2(\gamma_2) - 1)T_1^{r-1} + \dots + b_0(\psi_w(\gamma_2) - 1),$$

which clearly still has degree  $r = \deg_W(\mathcal{L}_p(\rho_0; T_1, T_2))$  in  $T_1$ . Let us therefore define  $n(0)$  to be the exponent of  $p$  in  $q(0)$ . It then follows that for any integer  $n \geq n(0)$ , we have the relation

$$(21) \quad \deg_W(\mathcal{L}_p(\rho_0; T_1^{(n)})) = [K_n : K] \cdot \deg_W(\mathcal{L}_p(\rho_1, T_1, T_2))$$

as elements in the base power series ring  $\mathcal{O}[[T_1]]$ . On the other hand, it is clear from the defining relation (19) (taking images under the isomorphism (17)) that the Weierstrass degree of  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  in the basechange power series ring  $\mathcal{O}[[T_1^{(n)}]]$  must be given by this quantity (21) divided by the index  $[K_n : K]$ , as required.  $\square$

Let us now consider the interpolation properties of these elements  $\mathcal{L}_p(\rho_0; T_1^{(n)})$ . That is, by Theorem 2.1, each basechange  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  in  $\mathcal{O}[[T_1^{(n)}]]$  must interpolate the associated basechange central values

$$L(1/2, \Pi_{f,n} \times \rho') = \prod_{\psi_w} L(1/2, f \times \rho_0 \rho_w \psi_w)$$

described in (14) above. Here again, we have taken the product over characters  $\psi_w = \chi \circ \mathbf{N}$  of the Galois group  $\text{Gal}(K_n/K)$ , and written  $\rho'$  to denote the finite order character of  $\Omega^{(n)}$  defined by the composition  $\rho \circ \mathbf{N}_{K_n/K}$ , where  $\rho = \rho_0 \rho_w$ . Now, recall that we have the root number relation (15) under basechange. This in particular allows us to deduce that for any ring class character  $\rho' = \rho_0 \rho_w \circ \mathbf{N}_{K_n/K}$  factoring through  $\Omega^{(n)}$ , the root number  $\epsilon(1/2, \Pi_{f,n} \times \rho')$  of the basechange  $L$ -function  $L(s, \Pi_{f,n} \times \rho')$  is given by the root number  $\epsilon(1/2, f \times \theta_K)$  of the base  $L$ -function  $L(s, f \times \theta_K)$ . In particular, if  $\epsilon(1/2, f \times \theta_K)$  equals  $-1$ , then it is easy to see from the functional equation satisfied by each of the (self-dual)  $L$ -functions  $L(s, \Pi_{f,n} \times \rho')$  that the basechange  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  must vanish identically in  $\mathcal{O}[[T_1^{(n)}]]$ . To surmount this issue, we now make the following modification in our construction of the basechange  $p$ -adic  $L$ -functions  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  in the setting where the root number  $\epsilon(1/2, f \times \theta_K)$  is  $-1$ . Thus, when  $\epsilon(1/2, f \times \theta_K)$  is  $-1$ , let us define the following incomplete basechange element

$$(22) \quad \mathcal{L}_p^*(\rho_0; T_1^{(n)}) = \prod_{\psi_w \neq \mathbf{1}} \mathcal{L}_p(\rho_0; T_1, \psi_w(\gamma_2) - 1) \in \mathcal{O}[[T_1]].$$

Here, the product runs over nontrivial characters  $\psi_w = \chi \circ \mathbf{N}$  of  $\text{Gal}(K_n/K)$ , and the product  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  is a priori only defined in the base power series ring  $\mathcal{O}[[T_1]]$ .

**Lemma 2.5.** *For each integer  $n \geq 1$ , the incomplete basechange element  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  of (22) above defines an element of the power series ring  $\mathcal{O}[[T_1^{(n)}]]$ . Moreover, the Weierstrass degree of  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  as an element of  $\mathcal{O}[[T_1^{(n)}]]$  is finite.*

*Proof.* Fix an integer  $n \geq 1$ . To show the first part of the claim, observe that each finite order character  $\rho_w^{(n)}$  of the Galois group  $\Omega^{(n)}$  defines a finite order character  $\rho_w$  of the Galois group  $\Omega^{(0)}$  by restriction to  $K$ . That is, each such character  $\rho_w^{(n)}$  arises as the composition of some finite order character  $\rho_w$  of  $\Omega^{(0)}$  with the norm homomorphism  $\mathbf{N}_{K_n/K}$ , as composition with  $\mathbf{N}_{K_n/K}$  induces a natural isomorphism of Galois groups  $\Omega^{(n)} \cong \Omega^{(0)}$ . This allows us to view  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  as an element of  $\mathcal{O}[[\Omega^{(n)}]]$ , whence taking its image under the isomorphism (17) defines an element of  $\mathcal{O}[[T_1^{(n)}]]$ . More precisely, each  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  defines an element  $\lambda^{(n)}$  say in the completed group ring  $\mathcal{O}[[\Omega^{(n)}]]$ , characterized as an  $\mathcal{O}$ -valued measure on  $\Omega^{(n)}$  via the following interpolation property: for each finite order character  $\rho_w^{(n)} = \rho_w \circ \mathbf{N}_{K_n/K}$  of  $\Omega^{(n)}$ , we have the identity

$$\rho_w^{(n)}(\lambda^{(n)}) = \prod_{\psi_w \neq \mathbf{1}} \eta(f, \rho_0 \rho_w \psi_w) \cdot \frac{L(1/2, f \times \rho_0 \rho_w \psi_w)}{8\pi^2 \langle f, f \rangle} \in \overline{\mathbf{Q}}_p.$$

Here, the product ranges over nontrivial characters  $\psi_w = \chi \circ \mathbf{N}$  of  $\text{Gal}(K_n/K)$ ,  $\rho_w$  as before denotes the underlying character of  $\Omega$ , and all other notations are the same as in Theorem 2.1. That  $\lambda^{(n)}$  defines a distribution on  $\Omega^{(n)}$  is then a formal consequence of the fact that the convolution measure on  $\Omega^{(0)}$  corresponding to the product  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  defines a distribution on  $\Omega^{(0)} \cong \Omega^{(n)}$ .

To show the second part of the claim, we appeal to the proof of Proposition 2.2 in the setting where the root number  $\epsilon(1/2, f \times \theta_K)$  is  $-1$ . This in particular allows us to deduce the nontriviality of  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  in  $\mathcal{O}[[T_1^{(n)}]]$  as an interpolation series for the associated complex values

$$L^*(1/2, \Pi_{f,n} \times \rho_n) := \prod_{\psi_w \neq \mathbf{1}} L(1/2, f \times \rho_0 \rho_w \psi_w).$$

In particular, using the result of Proposition 2.2 in this setting, we may then invoke the Weierstrass preparation theorem to deduce that  $\mathcal{L}_p^*(\rho_0; T_1^{(n)})$  has finite Weierstrass degree as an element of the basechange power series ring  $\mathcal{O}[[T_1^{(n)}]]$ .  $\square$

Let us now introduce the following basechange  $p$ -adic  $L$ -function in the setting where the root number  $\epsilon(1/2, f \times \theta_K)$  is  $-1$ . Consider the element  $\mathcal{L}_p^*(\rho_0, T_1^{(1)})$  defined in (22) above, which by Lemma 2.5 can be viewed as an element in the power series ring  $\mathcal{O}[[T_1^{(1)}]]$ . Let  $\mathbf{N}_{K_n/K_1}$  denote the norm homomorphism from  $K_n$  to  $K_1$ . Given  $n \geq 1$  an integer, let us then define an associated element

$$(23) \quad g_p(\rho_0; T_1^{(n)}) = \mathcal{L}_p^*(\rho_0; T_1^{(1)}) \circ \mathbf{N}_{K_n/K_1}$$

in the power series ring  $\mathcal{O}[[T_1^{(n)}]]$ , where the composition with  $\mathbf{N}_{K_n/K_1}$  is on the underlying completed group ring element  $\lambda^{(1)}$  in  $\mathcal{O}[[\Omega^{(1)}]]$  described for Lemma 2.5 above. Note that the collection of these elements  $g_p(\rho_0; T_1^{(n)})$  for all  $n \geq 1$  contains information about the specialization of the two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1, T_2)$  to each nontrivial character  $\psi_w$  of  $\Gamma$ . We also have the following analogous version of Lemma 2.4 for each  $g_p(\rho_0; T_1^{(n)})$  as a power series in  $\mathcal{O}[[T_1^{(n)}]]$ .



**Corollary 2.6.** *Fix a tamely ramified character  $\mathcal{W}_0 = \rho_0$ . Let  $n(0) \geq 0$  be the integer of Lemma 2.4 above. Then, for each integer  $n \geq n(0)$ , the Weierstrass degree of  $g_p(\rho_0; T_1^{(n)})$  as an element of  $\mathcal{O}[[T_1^{(n)}]]$  is equal to the Weierstrass degree of  $g_p(\rho_0; T_1^{(1)})$  as an element  $\mathcal{O}[[T_1^{(1)}]]$ .*

*Proof.* Fix an integer  $n \geq n(0)$ . We can assume without loss of generality that  $n \geq 2$ . By the argument of Lemma 2.4, it is clear that  $g_p(\rho_0; T_1^{(n)})$  has Weierstrass degree equal to

$$\varphi(p^n) \cdot \deg_W(\mathcal{L}_p(\rho_0; T_1, T_2)) = [K_n : K_1] ([K_1 : K] - 1) \cdot \deg_W(\mathcal{L}_p(\rho_0; T_1, T_2))$$

as an element in  $\mathcal{O}[[T_1]]$ , where  $\deg_W(\mathcal{L}_p(\rho_0; T_1, T_2))$  denotes the Weierstrass degree of the two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1, T_2)$  in  $R_2[[T_1]] = \mathcal{O}[[T_2]][[T_1]]$ . Now, we saw in Lemma 2.5 that  $\mathcal{L}_p^*(\rho_0; T_1^{(1)})$  defines an element in the power series ring  $\mathcal{O}[[T_1^{(1)}]]$ , of some finite Weierstrass degree  $d(1)$  say. Following the argument of Lemma 2.4, we deduce that the Weierstrass degree of  $\mathcal{L}_p^*(\rho_0; T_1^{(1)}) \circ \mathbf{N}_{K_n/K_1}$  in  $\mathcal{O}[[T_1^{(1)}]]$  is equal to  $p^{n-1}d(1) = [K_n : K_1]d(1)$ . This formula can be viewed as a consequence of the fact that specializations to characters of  $\text{Gal}(K_n/K_1)$  do not change the Weierstrass degree in  $T_1^{(1)}$ , or to be more precise the Weierstrass degree in  $T_1$  (as explained in the proof of Lemma 2.4). It is then a formal consequence that the Weierstrass degree of  $g_p(\rho_0; T_1^{(n)}) = \mathcal{L}_p^*(\rho_0; T_1^{(n)}) \circ \mathbf{N}_{K_n/K_1}$  in  $\mathcal{O}[[T_1^{(n)}]]$  is given by  $d(1)$ , or in other words by its Weierstrass degree in  $\mathcal{O}[[T_1^{(1)}]]$  divided by the index  $[K_n : K_1]$ .  $\square$

These results in particular allow us to make uniform the choice of minimal ring class conductor  $c(n)$  in Proposition 2.2 above. That is, we can now establish the following main result.

**Theorem 2.7.** *Fix a tamely ramified character  $\mathcal{W}_0 = \rho_0$  of  $G_0$ . Let  $n \geq 0$  be any integer. There exists an integer  $c(0) \geq 0$ , independent of choice of  $n$ , such that for all ring class conductors  $c \geq c(0)$ , the associated Galois average  $\delta_{c, p^n; \rho_0}^{(k)}$  does not vanish. Here,  $k = 0$  or  $1$  according as to whether or not the pair  $(f, \mathcal{W})$  is generic or exceptional.*

*Proof.* Recall the main result of Proposition 2.2 above, which asserts that for each integer  $n \geq 0$ , there exists an integer  $c(n) \geq 0$  such that for all  $(p$ -power) ring class conductors  $c \geq c(n)$ , the associated Galois average  $\delta_{c, p^n; \rho_0}^{(k)}$  does not vanish. We can assume without loss of generality that we are in the generic setting with  $k = 0$ , since otherwise the results is known by the relevant theorems of [3] and [4]. We now proceed by dividing into cases on the root number  $\epsilon(1/2, f \times \theta_K)$ , as in the proof of Proposition 2.2 above.

Let us first suppose that the root number  $\epsilon(1/2, f \times \theta_K)$  is  $+1$ . Consider the basechange  $p$ -adic  $L$ -functions  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  defined in (18) above. Starting with  $n = 0$ , we see from the Weierstrass preparation theorem that  $c(0)$  is determined by the Weierstrass degree  $\deg_W(0)$  of  $\mathcal{L}_p(\rho_0; T_1^{(0)}) = \mathcal{L}_p(\rho_0; T_1, 0)$  in  $\mathcal{O}[[T_1]]$ . More precisely, writing  $r(0)$  to denote the number of ring class characters  $\rho = \rho_0 \rho_w$  of conductor  $c \leq c(0)$  for which the Galois average  $\delta_{c, p^0; \rho_0}^{(0)}$  vanishes, it is easy to see from the interpolation property of Theorem 2.1 that  $r(0) = \deg_W(0)$ . Let us now fix an integer  $n \geq n(0)$ , where  $n(0) \geq 0$  is the integer of Lemma 2.4 above. Writing

$r(n)$  to denote the number of basechange ring class characters  $\rho' = \rho_0 \rho_w \circ \mathbf{N}_{K_n/K}$  of conductor  $c \leq c(n)$  for which the Galois average  $\delta_{c,p^n;\rho_0}^{(0)}$  vanishes, it then easy to see from the interpolation property of Theorem 2.1 along with Artin formalism for basechange values (14) that the Weierstrass degree  $\deg_W(n)$  of  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  in  $\mathcal{O}[[T_1^{(n)}]]$  is equal to  $r(n)$ . The result for  $n \geq n(0)$  then follows from that of Lemma 2.4, which shows that  $\deg_W(0) = \deg_W(n)$ , and hence that we can take  $c(n) = c(n(0))$  for each integer  $n \geq n(0)$  in the statement of Proposition 2.2. The result for integers  $n \leq n(0)$  then follows from another application of Artin formalism to the associated basechange central values  $L(1/2, \Pi_{f,n(0)} \times \rho')$ , where  $\rho' = \rho_0 \rho_w \circ \mathbf{N}_{K_{n(0)}/K}$  is any ring class character of conductor  $c \geq c(0)$ .

Let us now suppose that the root number  $\epsilon(1/2, f \times \theta_K)$  is  $-1$ , in which case we can assume without loss of generality that  $n \geq 1$  (since the case of  $n = 0$  corresponds to that of the exceptional setting treated by [3] and [4]). We can then use the same argument as given above for the case when the root number  $\epsilon(1/2, f \times \theta_K)$  is  $+1$ , replacing the basechange  $p$ -adic  $L$ -function  $\mathcal{L}_p(\rho_0; T_1^{(n)})$  (which vanishes identically by the associated functional equation) with the element  $g_p(\rho_0; T_1^{(n)})$  defined in (23) above (which does not vanish identically by the proof of Proposition 2.2), using Corollary 2.6 in lieu of Lemma 2.4 to obtain the required invariance of Weierstrass degree in the basechange variable  $T_1^{(n)}$  for integers  $n \geq n(0)$ .  $\square$

## REFERENCES

- [1] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over  $\mathbf{Q}$ : wild 3-adic exercises*, Journal of the American Mathematical Society **14** (2001), 843-939.
- [2] J. Coates, R. Sujatha, K. Kato and T. Fukaya, *Root numbers, Selmer groups and non-commutative Iwasawa theory*, Journal of Algebraic Geometry **19** (2010), 19 -97.
- [3] C. Cornut, *Mazur's conjecture on higher Heegner points*, Inventiones mathematicae **148** No. 3 (2002), 495 - 523.
- [4] C. Cornut and V. Vatsal, *Nontriviality of Rankin-Selberg  $L$ -functions and CM points*,  $L$ -functions and Galois Representations, Ed. Burns, Buzzard and Nekovář, Cambridge University Press (2007), 121- 186
- [5] B. Gross and D. Zagier, *Heegner points and derivatives of  $L$ -series*, Inventiones mathematicae **84** (1986), 225-320.
- [6] H. Hida, *A  $p$ -adic measure attached to the zeta functions associated to two elliptic modular forms I*, Inventiones mathematicae **79** (1985), 159-195.
- [7] B. Mazur and K. Rubin, *Elliptic Curves and Class Field Theory*, Proceedings of the ICM, Beijing 2002 Vol. 2, 185-196,
- [8] B. Perrin-Riou, *Fonctions  $L$ -adiques associées à une forme modulaire et à un corps quadratique imaginaire*, Journal of the London Mathematical Society (2) **38** (1988), 1 - 32.
- [9] D. Rohrlich, *On  $L$ -functions of elliptic curves and anticyclotomic towers*, Inventiones mathematicae **75** No. 3 (1984), 383 - 408.
- [10] D. Rohrlich, *On  $L$ -functions of elliptic curves and cyclotomic towers*, Inventiones mathematicae **75** No. 3 (1984), 409 - 423.
- [11] G. Shimura, *On the periods of modular forms*, Mathematische Annalen **229** No. 3 (1977), 211 - 221.
- [12] G. Shimura, *The special values of zeta functions associated with Hilbert modular forms*, Duke Mathematical Journal **45** (1978), 637-679.
- [13] C. Skinner and E. Urban, *The Main Conjecture for  $GL(2)$* , preprint (2010), available at <http://www.math.columbia.edu/~urban/EURP.html>
- [14] R. Taylor and A. Wiles, *Ring theoretic properties of certain Hecke algebras*, Annals of Mathematics **141** (1995), 553-572.

- [15] J. Van Order, *On the quaternionic  $p$ -adic  $L$ -functions associated to Hilbert modular eigenforms*, International Journal of Number Theory **8** No. 4 (2012), 1-35, available at <http://arxiv.org/abs/1112.3821>.
- [16] J. Van Order, *Rankin-Selberg  $L$ -functions in cyclotomic towers, I*, preprint (2012).
- [17] J. Van Order, *Rankin-Selberg  $L$ -functions in cyclotomic towers, III*, (preprint, 2012).
- [18] J. Van Order, *Some remarks on the two-variable main conjecture of Iwasawa theory for elliptic curves without complex multiplication*, Journal of Algebra **350** Issue 1 (2012), 273 - 299, available at <http://arxiv.org/abs/1110.6197>.
- [19] V. Vatsal, *Uniform distribution of Heegner points*, Inventiones mathematicae **148** No. 1 (2002), 1 - 46.
- [20] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Annals of Mathematics **141** (1995), 443-551.
- [21] X. Yuan, S.-W. Zhang, and W. Zhang, *Gross-Zagier Formula on Shimura Curves* (preprint), available at <http://www.math.columbia.edu/~szhang/papers/GZSC-2011-8.pdf>.
- [22] S.W. Zhang, *Gross-Zagier formula for  $GL_2$* , Asian Journal of Mathematics **5** No. 2 (2001), 183 - 290.

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